Online Appendix to:

“Flexible Information Acquisition in Large Coordination Games”

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C Determining the support of the signal and the message-to-action distribution

In order to determine the optimal signal support that a player i will use, let m_{-i} be any strategy profile that player i's opponents are using. Let s_i be the message that player i obtained after having chosen channel $\boldsymbol{s}_i$. Given s_i, player i forms a posterior belief about θ and from this belief and m_{-i} (by pushing forward), a posterior belief about the value of $\bar{a}$.[22] From these beliefs, player i forms expectations $\theta^i(s_i;\boldsymbol{s}_i)$ and $\bar{a}^i(s_i;\boldsymbol{s}_i,m_{-i})$ on the respective variables.

Lemma 2. *In a best response of player i, almost all messages $s_i \in S_i$ have the following property: there exists a unique action $a^{s_i} \in A_i$ such that* $\Pr(a^{s_i}|s_i) = 1$.

Proof. For simplicity of exposition it is assumed that $P_{s_i|\theta}$ is described by a PDF $q_i(\cdot|\theta)$ and $P_{a_i|s_i}$ is described by a PDF $\lambda_i(\cdot|s_i)$. The proof is essentially the same in the more general case.

Let $p^i(\cdot|s_i;\boldsymbol{s}_i)$ denote the PDF of the posterior belief that player i has about the fundamental (conditional on i receiving message s_i while using channel $\boldsymbol{s}_i$). It is calculated by Bayes's rule:

$$p^i(\theta|s_i;\boldsymbol{s}_i) = \frac{q_i(s_i|\theta)p(\theta)}{\int_\Theta q_i(s_i|\theta)\, p(\theta)\,\mathrm{d}\theta}.$$

Given player j's strategy and upon receiving message s_i, player i forms a posterior belief about j's message applying Bayes's rule once more. This is given by:

$$q^i_j(s_j|s_i;\boldsymbol{s}_i,\boldsymbol{s}_j) = \int_\Theta q_j(s_j|\theta)\, p^i(\theta|s_i;\boldsymbol{s}_i)\,\mathrm{d}\theta.$$

[22] Formally it should be $\bar{a}_{-i}$ (i.e. the average action of all players excluding player i) but as the contribution of a single player to the average action of a continuum of players is zero, $\bar{a}_{-i} = \bar{a}$.

So, player i's posterior belief about player j's action is

$$\lambda_j^i(a_j|s_i;\boldsymbol{s}_i,m_j)=\int_{S_j}\lambda_j(a_j|s_j)\,q_j^i(s_j|s_i;\boldsymbol{s}_i,m_j)\,\mathrm{d}s_j$$

and player i's expectation of player j's action is

$$a_j^i(s_i;\boldsymbol{s}_i,m_j)=\int_{A_j}a_j\,\lambda_j^i(a_j|s_i;\boldsymbol{s}_i,m_j)\,\mathrm{d}a_j.$$

Therefore, player i's expectation of the average action of her opponents is

$$\bar{a}_{-i}^i(s_i;\boldsymbol{s}_i,m_{-i})=\bar{a}^i(s_i;\boldsymbol{s}_i,m_{-i})=\int_0^1\int_{A_j}a_j\,\lambda_j^i(a_j|s_i;\boldsymbol{s}_i,m_j)\,\mathrm{d}a_j\,\mathrm{d}j$$

Notice that this expectation is equal to i's expectation over the population-wide average action $\bar{a}$ as player i's action cannot affect the mean action in a continuum population. Finally, player i's expectation of the value of the fundamental is

$$\bar{\theta}^i(s_i;\boldsymbol{s}_i)=\int_{\Theta}\theta p^i(\theta|s_i;\boldsymbol{s}_i)\,\mathrm{d}\theta.$$

Any costs player i has spent on acquiring information are sunk at the time she observes s_i. So, her expected utility at that point is calculated by

$$\mathbb{E}_i(u_i|s_i;\boldsymbol{s}_i,m_{-i})=$$
$$-(1-\gamma)\int_{\Theta}(a_i-\theta)^2p^i(\theta|s_i;\boldsymbol{s}_i,m_{-i})\,\mathrm{d}\theta-\int_{\Theta}\gamma(a_i-\bar{a}(\theta))^2p^i(\theta|s_i;\boldsymbol{s}_i,m_{-i})\,\mathrm{d}\theta.$$

Given that player i maximizes expected utility, any "best" action b_i that receives positive density in a best response of player i has to satisfy the following first order condition.

$$\begin{aligned}b_i(s_i;\boldsymbol{s}_i,m_{-i}) &= (1-\gamma)\int_{\Theta}\theta p^i(\theta|s_i;\boldsymbol{s}_i)\,\mathrm{d}\theta+\gamma\int_{\Theta}\bar{a}(\theta)p^i(\theta|s_i;\boldsymbol{s}_i)\,\mathrm{d}\theta\\ &= (1-\gamma)\theta^i(s_i;\boldsymbol{s}_i)+\gamma\bar{a}^i(s_i;\boldsymbol{s}_i,m_{-i})\end{aligned}$$

As long as the integrals appearing in the right-hand side of the above equation are well-defined, there is a unique value of b_i that satisfies the above condition. Therefore, a best response should put all probability to that action, *i.e.,* $\Pr(a^{s_i}|s_i)=1$ with a^{s_i} given by the above equation. □

Moreover, there should be a unique message that maps to each action.

Lemma 3. *In a best response of player i, almost all actions $a_i \in A_i$ have the following property: there exists a unique message $s^{a_i} \in S_i$ such that* $\Pr(a_i|s^{a_i}) = 1$.

Proof. Consider two strategies: $m_i = (\boldsymbol{s}_i, \boldsymbol{a}_i)$ under which each action has a unique message that maps to it (through $\boldsymbol{a}_i$) and $m_i' = (\boldsymbol{s}_i', \boldsymbol{a}_i')$ under which a set of actions of positive measure (under the measure induced on A_i by m_i') have multiple messages that map to them. For each action $a_i \in A_i$, denote by $S'(a_i)$ the set of messages that map to a_i under $\boldsymbol{a}_i'$ i.e. $S'(a_i) = \left\{s_i' \in S_i' : \Pr\left(\boldsymbol{a}_i = a_i|s_i'\right) = 1\right\}$ and by s^{a_i} the (unique) message that maps to a_i under $\boldsymbol{a}_i$. Note that $S'(a_i)$ should be nonempty for almost all a_i as a result of Lemma 2. Let also $q(s^{a_i}|\theta) = \sum_{s' \in S'(a_i)} q'(s|\theta)$. It is clear that the expected value of $-(1-\gamma)(a_i-\theta)^2 - \gamma(a_i-\bar{a})^2$ from the two strategies will be the same as they induce the same probability distribution on A_i for the same values of θ. It is also true that $I(\boldsymbol{\theta}, \boldsymbol{a}_i') > I(\boldsymbol{\theta}, \boldsymbol{a}_i)$ due to the convexity of mutual information in q (see Fozunbal, McLaughlin, and Schafer 2005). Therefore, m_i' is more expensive to player i than m_i and thus not an optimal choice. □

From Lemmas 2 and 3, the action part of the strategy $\boldsymbol{a}_i$ can be summarized by a bijection from S_i to A_i such that $\boldsymbol{a}_i$ gives probability one to a unique action a_i for each message $s_i \in S_i$, and for each action $a_i \in \mathbb{R}$ there exists a unique message for which $\Pr(a_i|s_i) = 1$. Thus, the message space should have the same cardinality as the action space. This should happen even if some of these messages are never used. Of course, if any of the messages is not to be used, this would immediately mean that the corresponding action would never be used by player i. So, player i's message space can be reduced to be a space equinumerous with $A_i = \mathbb{R}$. Since signals are important only as far as they prescribe probabilities over actions, the exact choice of the message space will not change players' actions as long as it has the same cardinality as $\mathbb{R}$, and $\boldsymbol{a}_i$ can be described by a bijection, as explained above.

D More General Payoffs

In this appendix a class of games broader than the one of the main text is considered. Propositions analogous to Propositions 2 and 6 are derived.

As in the main model, there is a large population of individuals. The decision problem facing each individual is a "tracking problem" à la Jung et al. (2019) but there are also strategic considerations. In particular, each individual i gets payoff that is given by

$$-u(a_i - b) - \tilde{u}(b, \theta)$$

where b is the "best action" and is given by $b := f(\bar{a}, \theta)$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$. The partial derivative $\partial_1 f$ is positive everywhere, so that the game is a coordination game. Moreover, the best action depends monotonically on the state of the world and so, without loss of generality, $\partial_2 f > 0$ everywhere.

The function $u : \mathbb{R} \to \mathbb{R}$ is assumed to be even, analytic, and $u'(x) < 0$ for $x < 0$ whereas $u'(x) > 0$ for $x > 0$. Moreover, $\int_{-\infty}^{+\infty} \exp(-u(x)/\mu)\,\mathrm{d}x < \infty$. Players' payoffs are also affected by $\tilde{u} : \mathbb{R}^2 \to \mathbb{R}$ but no individual player can affect its value. As $\tilde{u}$ will not enter the individuals' decision-making problem, no assumptions are imposed on it. This model includes beauty contests (analyzed in the main text) as well as all linear-quadratic large games as special cases. In particular, it includes all examples of Bergemann and Morris (2013) with infinite players and one-dimensional action sets.[23]

Individuals can acquire information about θ paying a cost linear in the reduction of Shannon entropy, as in the main model.

Best Response

A strategy profile of player i's opponents defines an average action function $\bar{a} : \mathbb{R} \to \mathbb{R}$ to which player i needs to best respond. The best action that player i needs to track is given by $B(\theta) := f(\bar{a}(\theta), \theta)$. A smooth, monotone, full-support profile of player i's opponents requires that B is strictly increasing, which imposes the condition that $\bar{a}'(\theta) > -\partial_1 f(\bar{a}(\theta), \theta)/\partial_2 f(\bar{a}(\theta), \theta)$ at all θ.

Slightly abusing notation, let $\theta(\cdot)$ denote the inverse of $B(\cdot)$. Since the fundamental is distributed according to the full-support density p, the best action is distributed according to

$$g(b) = p(\theta(b))\;\theta'(b)$$

[23] The beauty contest game of the main text is obtained through setting $f(\bar{a}, \theta) := \gamma \bar{a} + (1 - \gamma)\theta$ and $u(x) := x^2$.

which has full support. Let also $z : \mathbb{R} \to \mathbb{R}_{>0}$ be defined through

$$z(x) := \frac{\exp(-u(x)/\mu)}{\int_{-\infty}^{+\infty} \exp(-u(x)/\mu)\, \mathrm{d}x}.$$

A generalization of Proposition 2 can now be formulated (see also Jung et al. 2019, Proposition 2 for this result).

Proposition 12. *Player i has a continuous best response to a smooth monotone full-support profile of her opponents if and only if*

$$R_i := \mathscr{F}^{-1}[\hat{g}/\hat{z}] \qquad \textit{is the PDF of a probability distribution.} \tag{50}$$

This continuous strategy is her unique best response and is given by

$$r_i(a_i|b) = R_i(a_i)\, \frac{z(a_i - b)}{g(b)}.$$

where $R_i(a_i)$ is the marginal density of action a_i.

Corollary. *If player i's best response is continuous, her posterior belief about the best action is given by*

$$\tau(b|a_i) = z(a_i - b).$$

This is the analogue of Proposition 3. Notice that this posterior belief is fully determined by the payoff function $u(\cdot)$ and the cost μ. In particular, it does not depend on the best action's distribution $g(\cdot)$ (or, for that matter $p(\cdot)$) nor does it depend on how this best action is being calculated (the particular function $f(\cdot)$).

Equilibrium

The characterization of SMFE in the general case — analogous to Proposition 6 — is provided below.

Proposition 13. *The following two statements are equivalent*

(A) *$\theta(\cdot)$ is the inverse of the best action function and $g(\cdot)$ is the PDF of the distribution of the best action in an SMFE.*

(B) $\theta : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijection, $\mathscr{F}^{-1}[\hat{g}/\hat{z}]$ is a probability distribution,

$$b = f\left(b - \frac{(g * Z)(b)}{g(b)}, \theta(b)\right) \quad \text{and} \tag{51}$$

$$g(b) = p(\theta(b))\theta'(b). \tag{52}$$

Where $Z(x) := x\mathscr{F}^{-1}[\log \circ \hat{z}](x)$.

E Proofs

E.1 Proof of Proposition 12

Following essentially the same process as in the proof of Proposition 2, one calculates the best response of player i to g:

$$r_i(a_i|b) = \exp\left(\frac{-\tilde{u}(b, \theta_{\tilde{a}}(b))}{\mu}\right)\exp\left(-\frac{k(b)}{\mu}\right)R(a_i)\exp\left(\frac{-u(a_i - b)}{\mu}\right)$$

where $k(b)$ ($b \in \mathbb{R}$) are Lagrange multipliers. In order to have a continuous best response, the solution should also satisfy the constraints $\int_{-\infty}^{+\infty} r(a_i|b)g(b)\,\mathrm{d}b = R(a_i)$ for all $a_i \in \mathbb{R}$, which gives

$$\int_{-\infty}^{+\infty} \underbrace{\exp\left(\frac{-\tilde{u}(b, \theta_{\tilde{a}}(b))}{\mu}\right)\exp\left(-\frac{k(b)}{\mu}\right)g(b)}_{G(b)}\exp\left(\frac{-u(a_i - b)}{\mu}\right)\mathrm{d}b = 1$$

or, simply,

$$\int_{-\infty}^{+\infty} G(b)\ \exp(-u(a_i - b)/\mu)\,\mathrm{d}b = 1 \Rightarrow (G * \exp(-u/\mu))(a_i) = 1 \qquad \text{for all } a_i \in \mathbb{R}.$$

Using the convolution theorem, and after calculations, one gets that

$$G(b) = \left(\int_{-\infty}^{+\infty} \exp(-u(x)/\mu)\,\mathrm{d}x\right)^{-1} =: G,$$

i.e., a normalizing constant. Using the definition of z, i.e., that $z(x) = G\exp(-u(x)/\mu)$, the best response of player i can be written as

$$r_i(a_i|b) = \frac{R(a_i)}{g(b)}z(a_i - b).$$

All that remains is for R to be determined. This comes from the constraint $\int_{-\infty}^{+\infty} r(a_i|b)\,\mathrm{d}a_i = 1$ for (almost) all b which, in turn, yields

$$\int_{-\infty}^{+\infty} z(a_i - b)R(a_i)\,\mathrm{d}a_i = g(b) \Rightarrow (R * z)(b) = g(b),$$

as z is even. This means that $\hat{R} = \hat{g}/\hat{z}$. So, if g can be deconvolved with z as a kernel, then the problem has a solution in continuous strategies with best response given by

$$r_i(a_i|b) = \mathscr{F}^{-1}[\hat{g}/\hat{z}](a_i)\,\frac{z(a_i - b)}{g(b)}.$$

□

E.2 Proof of Proposition 13

Begin by calculating $\alpha(b)$, the expected action of player i conditional on the best action being b when she is best-responding. This can be calculated using the property:

$$\alpha(b) = \frac{1}{-2\pi\imath}\,(\mathscr{F}_a[r(a|b)])'\,(0)$$

Calculations give:

$$\begin{aligned}
\mathscr{F}_a[r(a|b)](\xi) &= \mathscr{F}_a\left[\frac{R(a)z(a-b)}{g(b)}\right](\xi) = \frac{1}{g(b)}\mathscr{F}_a[R(a)z(a-b)](\xi) \\
&= \frac{1}{g(b)}(\hat{R} * \mathscr{F}_a[z(a-b)])(\xi) = \frac{1}{g(b)}\left(\hat{R} * \exp(-2\pi\imath b\xi)\hat{z}\right)(\xi) \\
&= \frac{1}{g(b)}\int_{-\infty}^{+\infty}\frac{\hat{g}(y)}{\hat{z}(y)}\exp(-2\pi\imath b(\xi - y))\hat{z}(\xi - y)\,\mathrm{d}y
\end{aligned}$$

and the first derivative at $\xi = 0$ gives:

$$(\mathscr{F}_a[r(a|b)])'\,(0) = \frac{1}{g(b)}\int_{-\infty}^{+\infty}\frac{\hat{g}(y)}{\hat{z}(y)}\exp(2\pi\imath b y)\left(\hat{z}'(-y) - 2\pi\imath b\hat{z}(-y)\right)\mathrm{d}y$$

As z is an even, purely real function, $\hat{x}$ is also even and purely real. thus $\hat{z}'$ is a purely real, odd function. So,

$$(\mathscr{F}_a[r(a|b)])'\,(0) = \frac{1}{g(b)}\left(-\mathscr{F}^{-1}\left[\hat{g}(y)(\log\circ\hat{z})'(y)\right](b) - 2\pi\imath b g(b)\right)$$

and, finally,

$$\alpha(b) = b - \frac{(g * Z)(b)}{g(b)}$$

where $Z(x) := x\mathscr{F}^{-1}[\log \circ \hat{z}](x)$. Notice that Z is fully determined by the payoff function u and the information cost μ.

Having calculated α (as was done in Proposition 4), the rest of the proof is identical to the proof of Proposition 6, using

$$b = f(\alpha(b), \theta(b))$$

as the fixed point condition. □

References

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